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On functional equations of local zeta functions of
prehomogeneous vector spaces

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Introduction

In 1960's, M.Sato introduced the notion of prehomogeneous vector spaces and proved the functional equations of zeta functions associated with prehomogeneous vector spaces defined over \mathbb{R} or \mathbb{C}

$$(|f(x)|_i^s)^{\wedge} = \sum_j \gamma_{ij}(s) |f(x)|_j^{-n/d-s},$$

where $f(x)$ is the relative invariant of the prehomogeneous vector space V , $\gamma_{ij}(s)$ are meromorphic functions on \mathbb{C} , $n = \dim V$, d is the degree of $f(x)$, \wedge means the Fourier transform, and

$$|f(x)|_j^{-n/d-s} := \begin{cases} |f(x)|^{-n/d-s} & x \in V_j \\ 0 & \text{otherwise} \end{cases}$$

where V_j are G_{κ} -orbits in $V-S$ (for detail, see [SS]).

Similar functional equations associated with regular prehomogeneous vector spaces defined over k -adic fields have been proved by J.Igusa and F.Sato assuming some conditions of its singular set ([S],[I]).

But even when a prehomogeneous vector space (G, ρ, V) satisfies the sufficient conditions of F.Sato which assure the functional equations of zeta functions, the prehomogeneous vector space $(\tilde{G}, \tilde{\rho}, \tilde{V})$, which is obtained by the castling transform of (G, ρ, V) , does not necessarily satisfy them(cf.[S],[SK]).

Thus even if the functional equations of zeta functions of (G, ρ, V) hold, we do not know whether the functional equations of zeta

functions of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ hold or not. Since the castling transform is a standard procedure of constructing new prehomogeneous vector spaces, it is natural to ask the existence of the functional equations of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ when the functional equations of (G, ρ, V) hold.

In this paper we prove the following theorem using the results of [S0]:

Theorem *If the functional equations of zeta functions of (G, ρ, V) hold, then the functional equations of zeta functions of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ hold and vice versa (see §4).*

§1 Preliminaries

First we recall some basic notions in the theory of prehomogeneous vector spaces over a k -adic field and give some definitions following [S0].

Let k be a k -adic field and denote by \bar{k} (resp. \mathcal{O}_k) its algebraic closure (resp. maximal order). Let G be a connected linear algebraic group defined over k and V be a finite dimensional \bar{k} -vector space with k -structure V_k . Let $\rho: G \rightarrow GL(V)$ be a k -rational representation of G on V . Then the triple (G, ρ, V) is called a *prehomogeneous vector space* if there exists a proper algebraic subset S of V such that $V-S$ is a single $\rho(G)$ -orbit. The algebraic set S is called the *singular set*. It is known that $V-S$ is a single $\rho(G)$ -orbit implies that the number of $\rho(G_k)$ -orbits in V_k-S_k is finite. A nonzero rational function $P(v)$ is called a *relative invariant* of (G, ρ, V) if there exists a rational character $\chi(g)$ of G such that

$$P(\rho(g)v) = \chi(g)P(v) \quad (g \in G, v \in V).$$

Let S_i ($1 \leq i \leq l$) be the k -irreducible hypersurfaces contained in S . For each i ($1 \leq i \leq l$), take a k -irreducible polynomial $P_i(v)$ defining S_i . Then it is known that $P_i(v)$ are relative invariants and any relative invariant $P(v)$ in $k[V]$ is written uniquely as

$$P(v) = c \prod_{i=1}^l P_i(v)^{v_i} \quad (c \in k^\times, v_1, \dots, v_l \in \mathbb{Z}).$$

The polynomials P_1, \dots, P_l are called the *basic relative invariants* of (G, ρ, V) , and l the k -rank of (G, ρ, V) .

A relative invariant $P(v)$ is called *nondegenerate* if the Hessian $\det\left(\frac{\partial^2 P}{\partial v_i \partial v_j}\right)$ does not vanish identically. A prehomogeneous vector space (G, ρ, V) is called *regular* if there exists a nondegenerate relative invariant; and then one can find a nondegenerate relative invariant in $k[V]$.

Let V^* be the vector space dual to V and $\rho^* : G \rightarrow GL(V^*)$ the rational representation of G contragredient to ρ . The vector space V^* has a k -structure canonically defined by the k -structure of V . Then the representation ρ^* is defined over k .

Let m and n be positive integers with $m > n \geq 1$. We consider a rational representation $\rho_0 : H \rightarrow GL(m)$ of a connected linear algebraic group H . We assume that H and ρ_0 are defined over the field k . Put $G = H \times GL(n)$ and $V = M(m, n)$. Also put $\tilde{G} = H \times GL(m-n)$ and $\tilde{V} = M(m, m-n)$. Let $\rho : G \rightarrow GL(V)$ (resp. $\tilde{\rho} : \tilde{G} \rightarrow GL(\tilde{V})$) be a rational representation of G on V (resp. \tilde{G} on \tilde{V}) defined by

$$\rho(h, g_n)v = \rho_0(h)vg_n^{-1} \quad ((h, g_n) \in G = H \times GL(n))$$

$$(\text{resp. } \tilde{\rho}(h, g_{m-n})u = {}^t\rho_0(h)^{-1}u {}^tg_{m-n} \quad ((h, g_{m-n}) \in \tilde{G} = H \times GL(m-n))).$$

The triple $(\tilde{G}, \tilde{\rho}, \tilde{V})$ is called the *castling transform* of (G, ρ, V) and vice versa. Then we have the following lemma:

Lemma 1 (Sato-Kimura [SK1]) *The triplet (G, ρ, V) is a prehomogeneous vector space if and only if so is its castling transform $(\tilde{G}, \tilde{\rho}, \tilde{V})$.*

In the following, we assume that (G, ρ, V) and $(\tilde{G}, \tilde{\rho}, \tilde{V})$ are prehomogeneous vector spaces with k -structure

$$\begin{aligned} G_k &= H_k \times GL(n; k), & V_k &= M(m, n; k), \\ \tilde{G}_k &= H_k \times GL(m-n; k), & \tilde{V}_k &= M(m, m-n; k). \end{aligned}$$

Put $N = \binom{m}{n}$ and let $\Delta_1(v), \dots, \Delta_N(v)$ (resp. $\tilde{\Delta}_1(w), \dots, \tilde{\Delta}_N(w)$) be the minor determinants of $v \in V$ (resp. $w \in \tilde{V}$) of size n (resp. $m-n$).

Let V_0 be the vector space of column vectors of m entries and V_0^* the vector space dual to V_0 . We identify V (resp. \tilde{V}) with the direct product of n (resp. $m-n$) copies of V_0 (resp. V_0^*) in the standard manner. Let $\Delta : V \longrightarrow \bigwedge^n V_0$ and $\tilde{\Delta} : \tilde{V} \longrightarrow \bigwedge^{m-n} V_0^*$ be the mapping defined by

$$\Delta(v) = \Delta(v_1, \dots, v_n) = v_1 \wedge \dots \wedge v_n \quad (v_1, \dots, v_n \in V_0)$$

and

$$\tilde{\Delta}(w) = \tilde{\Delta}(w_1, \dots, w_{m-n}) = w_1 \wedge \dots \wedge w_{m-n} \quad (w_1, \dots, w_{m-n} \in V_0^*),$$

respectively. We identify $\bigwedge^n V_0$ with $\bigwedge^{m-n} V_0^*$ via the canonical isomorphism:

$$\bigwedge^n V_0 \xrightarrow{\simeq} (\bigwedge^{m-n} V_0)^* \xrightarrow{\simeq} \bigwedge^{m-n} V_0^*.$$

By taking the standard basis, we may identify $\bigwedge^n V_0$ and $\bigwedge^{m-n} V_0^*$ with k^N , so that the mappings Δ and $\tilde{\Delta}$ are given by

$$\begin{aligned}\Delta(v) &= (\Delta_1(v), \dots, \Delta_N(v)) \\ \tilde{\Delta}(w) &= (\tilde{\Delta}_1(w), \dots, \tilde{\Delta}_N(w)).\end{aligned}$$

Here the minor determinants are indexed such that

$$\det(v, w) = \sum_{i=1}^N \Delta_i(v) \tilde{\Delta}_i(w).$$

Then it is easy to see that

$$\Delta(\rho(h, g_n)v) = \det g_n^{-1} \cdot \left(\bigwedge^n \rho_0(h) \right) (\Delta(v))$$

and

$$\tilde{\Delta}(\tilde{\rho}(h, g_{m-n})w) = (\det g_{m-n} / \det \rho_0(h)) \cdot \left(\bigwedge^n \rho_0(h) \right) (\tilde{\Delta}(w)).$$

Thus we consider that G (resp. \tilde{G}) operates on $\Delta(V)$ (resp. $\Delta(\tilde{V})$).

Now put

$$\begin{aligned}V' &= \{v \in V \mid \text{rank } v = n\} \quad (\text{resp. } \tilde{V}' = \{w \in \tilde{V} \mid \text{rank } w = m-n\}) \\ V'_k &= V' \cap V_k \quad (\text{resp. } \tilde{V}'_k = \tilde{V}' \cap \tilde{V}_k).\end{aligned}$$

Then we have

$$\begin{aligned}Y &= \Delta(V') = \tilde{\Delta}(\tilde{V}') \subset \bar{k}^N \\ Y_k &= \Delta(V'_k) = \tilde{\Delta}(\tilde{V}'_k) \subset k^N.\end{aligned}$$

Moreover we have the following lemma:

Lemma 2 (cf. [S0] Lemma 1.2)

- (1) The k -rank of (G, ρ, V) is equal to the k -rank of $(\tilde{G}, \tilde{\rho}, \tilde{V})$.
- (2) There exist irreducible homogeneous polynomials $Q_1, \dots, Q_l \in k[y_1, \dots, y_n]$ ($l = \text{the } k\text{-rank of } (G, \rho, V)$) such that

$$P_1(v) = Q_1(\Delta(v)), \dots, P_l(v) = Q_l(\Delta(v))$$

are the basic relative invariants of (G, ρ, V) over k and

$$\tilde{P}_1(w) = Q_1(\tilde{\Delta}(w)), \dots, \tilde{P}_l(w) = Q_l(\tilde{\Delta}(w))$$

are the basic relative invariants of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ over k .

- (3) Put $d_i = \deg Q_i$. Then there exist k -rational characters ψ_1, \dots, ψ_l of H such that

$$\begin{aligned}
P_i(\rho(h, g_n)v) &= \chi_i(h, g_n)P_i(v), \quad \chi_i(h, g_n) = (\det g_n)^{-d_i} \cdot \psi_i(h) \\
\tilde{P}_i(\tilde{\rho}(h, g_{m-n})w) &= \tilde{\chi}_i(h, g_{m-n})\tilde{P}_i(w), \\
\tilde{\chi}_i(h, g_{m-n}) &= (\det g_{m-n} / \det \rho_0(h))^{d_i} \cdot \psi_i(h).
\end{aligned}$$

Let X be a subset of V_k (resp. \tilde{V}_k) to which G_k (resp. \tilde{G}_k) operates. Denote by $\mathcal{G}(X)$ the Schwartz-Bruhat space of the topological space X and by $\mathcal{G}'(X)$ its dual space as a \mathbb{C} -vector space. We call an element of $\mathcal{G}'(X)$ a *distribution* on X . And for a subset X' of X , we denote the characteristic function of X' by $Ch(X')$. For $f \in \mathcal{G}(X)$, $T \in \mathcal{G}'(X)$, $g \in G_k$, we define $f^g(x) \in \mathcal{G}(X)$ and $gT \in \mathcal{G}'(X)$ as follows:

$$\begin{aligned}
f^g(x) &= f(\rho(g)x), \\
gT(f(x)) &= T(f^g(x)) = T(f(\rho(g)x)).
\end{aligned}$$

In this paper, we always mean by a character ω of G_k a character of G_k of the form

$$\omega = \phi(\chi),$$

where $\chi: G \rightarrow k^\times$ is a k -rational character of G , and $\phi: k^\times \rightarrow \mathbb{C}^\times$ is a continuous homomorphism.

Now for a group G_k and a character $\omega: G_k \rightarrow \mathbb{C}^\times$, define $\xi(X, G_k, \omega)$ as follows:

$$\xi(X, G_k, \omega) = \{T \in \mathcal{G}'(X) \mid gT = \omega(g)^{-1}T \text{ for all } g \in G_k\}.$$

And for a subgroup H of G_k we define the *modular function* $\delta(H)$ by $d(h_0^{-1}hh_0) = \delta(H)(h_0) \cdot dh$, where dh is a left invariant measure on H .

Now we quote the following lemma which will be used in the next chapter:

Lemma 3 ([I] p.1015) *Let G be a linear algebraic group defined over*

k , G_k be the set of its k -rational points and H_k be a closed subgroup of G_k . Let $x = G_k/H_k$, and ω as above; then $\xi(x, G_k, \omega) \neq 0$ if and only if

$$\omega \cdot \delta(G_k)|_{H_k} = \delta(H_k),$$

and in that case $\dim_{\mathbb{C}} \xi(x, G_k, \omega) = 1$

Put

$$S(1) = \{v \in V_k \mid \text{rank } v < n\}.$$

Then we have

$$\rho(G_k)S(1) = S(1).$$

Thanks to results of [S0], we know that our theorem is valid if we restrict ourselves to the case where $x \in V_k \setminus S(1)$. Therefore we study the distributions $\xi(S(1), G_k, \omega)$ in §2 and prove the main theorem in §3.

§2 We keep the notations in §1. The aim of this section is to prove the following proposition:

Proposition 1

If $\xi(S(1), G_k, \omega) \neq 0$, there exist a finite number of subgroups H_i , $1 \leq i \leq q$ of $GL(n; k)$ such that

$$\omega|_{H_i} = \delta(H_i) \quad \text{for some } i,$$

and $\delta(H_i)$ is not identically equal to 1 if $H_i \neq GL(n; k)$.

Let V_k be as before and G' the set of k -rational points of a connected algebraic group which acts on V_k by a k -rational representation ρ .

Let A be a G' -stable subset of V_k such that for any $x, y \in A$, the isotropy subgroups G'_x and G'_y are conjugate in G' . Then there exists a complete system V_A of representatives of G' -orbits in A such that $G'_x = G'_y$ for any $x, y \in V_A$. We put $G'_A = G'_x$ for $x \in V_A$. Then we have a bijection

$$\Phi: G'/G'_A \times V_A \longrightarrow A$$

defined by $\Phi(g \cdot G'_A, x) = \rho(g)x$.

Now we assume that

Φ is a homeomorphism.

Lemma 4 *Under the conditions above, we have*

$$\xi(A, G', \omega) \neq 0 \text{ if and only if } \omega \cdot \delta(G')|_{G'_{iso}} = \delta(G'_{iso}).$$

Proof. It is easy to see that if there exists a G' -orbit \emptyset in A such that $\xi(\emptyset, G', \omega) \neq 0$, then we have $\xi(A, G', \omega) \neq 0$. Therefore "if" part is trivial from Lemma 3. Now we prove "only if" part.

Notice that any compact open subset of totally disconnected topological space $G'/G'_A \times V_A$ can be written as a disjoint union of compact open subsets of the form $U_o \times U_v$, where U_o (resp. U_v) is a compact open subset of G'/G'_A (resp. V_A).

Thus, if $\xi(A, G', \omega) \neq 0$, then there exists

$$T \in \xi(A, G', \omega) \text{ such that } T(Ch(\Phi(U))) \neq 0$$

where $U = U_o \times U_v$. We fix these U_o and U_v . Now for each compact

open subset $U(\sigma)$ of G'/G'_A , we define a mapping τ by

$$\tau: U(\sigma) \longrightarrow \Phi(U(\sigma) \times U_\nu).$$

Then we have

$$\tau(U_\sigma) = \Phi(U).$$

It is easy to see that τ is G' -admissible i.e.,

$$\tau(g \cdot U(\sigma)) = \Phi(g \cdot U(\sigma) \times U_\nu) = g \cdot \Phi(U(\sigma) \times U_\nu) = g \cdot \tau(U(\sigma)). \quad (1)$$

Now τ induces a linear mapping

$$\bar{\tau}: \mathcal{G}(G'/G'_A) \longrightarrow \mathcal{G}(A).$$

Using (1), we have

$$g \cdot (\bar{\tau}(f(x))) = \bar{\tau}(f(\rho(g)x))$$

for all $f \in \mathcal{G}(x)$. Now we define $F \in \mathcal{G}'(G'/G'_A)$ by

$$F(f) = T(\bar{\tau}(f)).$$

Then we have

$$\begin{aligned} g \cdot F(f) &= T(\bar{\tau}(f^g)) = T((\bar{\tau}(f))^g) \\ &= g \cdot T(\bar{\tau}(f)) = \omega^{-1}(g) T(\bar{\tau}(f)) = \omega(g)^{-1} F(f), \end{aligned}$$

where $g \in G'$.

Therefore we have $F \in \xi(G'/G'_A, G', \omega)$. Moreover we have

$$F(f) = T(Ch(U(\sigma))) \neq 0$$

for $f = Ch(U(\sigma))$, which implies

$$\xi(G'/G'_A, G', \omega) \neq 0.$$

Thus, from Lemma 3, we have

$$\omega \cdot \delta(G')|_{G'_A} = \delta(G'_A).$$

□

Now put

$$S_r = \{v \in S(1) \mid \text{rank } v = r\} \quad \text{for } 0 \leq r \leq n-1.$$

Then $\rho(1, GL(n; k)) S_r = S_r$ for all $0 \leq r \leq n-1$ and

$$S(1) = S_{n-1} \cup \dots \cup S_0 \quad (\text{disjoint union}).$$

Now the following lemma holds.

Lemma 5

Each S_r can be decomposed as follows:

$$S_r = \rho(1, GL)S_{r1} \cup \dots \cup \rho(1, GL)S_{rl} \quad (\text{disjoint union}),$$

where $l = \binom{m}{r}$, $GL = GL(n; k)$, and by putting $G' = GL$, $A = \rho(1, GL)S_{rh}$ and $V_A = S_{rh}$, these G' , A , and V_A satisfy the conditions (a), (b), and (c) above for all $1 \leq h \leq l$.

Proof. For a matrix $x = (x_{ij})$ and $j = 1, \dots, n$, we denote by $I(j, x)$ the smallest i for which $x_{ij} \neq 0$. We denote by $Rep(r)$ the set of matrices of the form

$$x = \begin{pmatrix} w & 0 \end{pmatrix} \in M(m, n), \quad w = (w_{ij}) \in M(m, r)$$

with the condition that

$$w_{I(j, x)h} = \delta_{jh} \quad (\text{Kronecker's symbol}) \quad \text{for all } 1 \leq j \leq r, 1 \leq h \leq r \quad \text{and} \\ I(j, x) \leq I(j+1, x) - 1 \quad \text{for all } 1 \leq j \leq r-1.$$

It is trivial that $Rep(r)$ is a complete system of representatives of $\rho(1, GL)$ -orbits in S_r . Also define $\alpha(x) \in \mathbb{Z}^r$ by

$$\alpha(x) = (I(1, x), \dots, I(r, x)).$$

Then, for $x \in Rep(r)$, we have

$$1 \leq I(1, x) < \dots < I(r, x) \leq m.$$

Now put

$$I = \{(i_1, \dots, i_r) \mid 1 \leq i_1 < \dots < i_r \leq m\}.$$

It is clear that $|I| = \binom{m}{r}$, and we number the elements of the set I in an arbitrary order and write I as follows:

$$I = \{I_i \mid (1 \leq i \leq \binom{m}{r})\}.$$

For any $I_t = (I(1), \dots, I(r)) \in I$ we define S_{ri} by

$$S_{ri} = \alpha^{-1}(I_t).$$

Now S_{ri} is homeomorphic to the k -vector space of dimension

$$(n - I(r)) \times r + \sum_{j=1}^{r-1} (I(j+1) - I(j) - 1) \times j,$$

and it is easy to see that if we put $G' = GL$, $A = \rho(1, GL)S_{rh}$, $V_A = S_{rh}$, these G' , A , V_A satisfy the conditions (a), (b), and (c). Moreover the condition

$$S_r = \rho(1, GL)S_{r1} \cup \dots \cup \rho(1, GL)S_{rl} \quad (\text{disjoint union})$$

is now trivial. □

Now we can prove Proposition 1.

Proof of Proposition 1. We keep the notation in Lemma 5.

Since $\xi(S(1), G_k, \omega) \neq 0$ implies $\xi(S(1), GL(n; k), \omega) \neq 0$, we have only to prove that the conditions in our proposition are necessary if $\xi(S(1), GL(n; k), \omega) \neq 0$.

The isotropy subgroup of $GL(n; k)$ for $x \in \text{Rep}(r)$, which we denote by H_x , is given by

$$H_x = \left\{ \begin{pmatrix} I_r & 0 \\ * & * \end{pmatrix} \in GL(n; k) \right\}, \text{ where } I_r \text{ is the identity matrix of size } r.$$

Therefore $\delta(H_x)$ is not identically equal to 1 if $H_x \neq GL(n; k)$

($H_x = GL(n; k)$ if and only if $x = 0$). Thus using Lemma 4 and Lemma 5,

$S(1)$ is decomposed as follows:

$$S(1) = \cup S_{rh} \quad (\text{disjoint union}),$$

where the union runs through $0 \leq r \leq n-1$, $1 \leq h \leq \binom{n}{r}$, and

$\rho(1, GL(n; k))S_{rh} = S_{rh}$ for all $1 \leq r \leq n-1$, $1 \leq h \leq \binom{n}{r}$, and if

$\xi(S_{rh}, GL(n; k), \omega) \neq 0$, then there exist $H_{rh} (= H_x \text{ for } x \in \text{Rep}(r)) \subset GL(n; k)$

such that

$$\delta(GL(n; k)) \cdot \omega|_{H_{rh}} = \delta(H_{rh})$$

and $\delta(H_{rh})$ is not identically equal to 1 if $H_{rh} \neq GL(n; k)$. Now since $GL(n; k)$ is unimodular, we have

$$\omega|_{H_{rh}} = \delta(H_{rh}).$$

Thus we have proved the proposition. \square

§3 Main result

For $\sigma = (\sigma_1, \dots, \sigma_l) \in \mathbb{C}^l$, we define the character ω_σ of G_k by

$$\omega_\sigma = (\phi_1 \omega_{\sigma_1}(\chi_1), \dots, \phi_l \omega_{\sigma_l}(\chi_l)),$$

where $\omega_{\sigma_i}(\alpha) = |\alpha|_k^{\sigma_i}$ ($\alpha \in k^\times$, $1 \leq i \leq l$), ϕ_i ($1 \leq i \leq l$) are dual of \mathcal{O}_k^\times and l

is the k -rank of (G, ρ, V) . Also define $\tilde{\omega}_\sigma$ for $\sigma = (\sigma_1, \dots, \sigma_l)$ by

$$\tilde{\omega}_\sigma = (\phi_1 \omega_{\sigma_1}(\tilde{\chi}_1), \dots, \phi_l \omega_{\sigma_l}(\tilde{\chi}_l)),$$

and put $\Omega = \{\sigma \in \mathbb{C}^l \mid \xi(S, G_k, \omega_\sigma) \neq 0\}$.

Now we recall some facts in the theory of prehomogeneous vector spaces roughly.

The fundamental theorem in the theory of regular prehomogeneous vector spaces states that there exist $u \in GL(n; \mathbb{Z})$, $\lambda \in \mathbb{C}^l$, and meromorphic functions $\gamma_{ij}(\sigma)$ such that

$$F(x, \sigma)_l = (|P(x)|_{k, l}^\sigma)^\wedge - \sum_j^v \gamma_{ij}(\sigma) \cdot |P^*(x)|_{k, j}^{\lambda + u\sigma}$$

vanishes for all $x \in V_k$, $\sigma \in \mathbb{C}^l$, and $1 \leq i, j \leq v$, i.e.,

$$(|P(x)|_{k, l}^\sigma)^\wedge = \sum_j^v \gamma_{ij}(\sigma) \cdot |P^*(x)|_{k, j}^{\lambda + u\sigma}, \quad (2)$$

where $^\wedge$ means the Fourier transform, v is the number of

$\rho(G_k)$ -orbits in $V_k - S_k$, and $|P(x)|_{k,i}^{\sigma} = \prod_{h=1}^l |P_h(x)|_{k,i}^{\sigma_h}$ (functional equations of zeta functions, see [S]).

For $x \in V_k - S$, the theorem above is proved by the uniqueness of relatively invariant distributions of homogeneous space (cf. Lemma 3). Moreover, thanks to results of [S0], we know that if the functional equations of zeta functions of (G, ρ, V) hold for $x \in V_k - S(1)$, then the functional equations of zeta functions of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ hold for $x \in \tilde{V}_k - \tilde{S}(1)$. On the other hand, it is known that $F(x, \sigma)_i \neq 0$ for some $x \in S(1)$ implies that there exists non-zero $T \in \xi(S(1), G_k, \omega_{\sigma})$ such that T is meromorphic with respect to $\sigma \in \mathbb{C}^l$. Therefore we can prove $F(x, \sigma)_i = 0$ for all $x \in S(1)$, $\sigma \in \mathbb{C}^l$ by showing that $\mathbb{C}^l \setminus \Omega$ is dense in \mathbb{C}^l . Hence, for the proof of the functional equations, it is enough to show that $\mathbb{C}^l \setminus \Omega$ is dense in \mathbb{C}^l .

Now we prove the main theorem.

Theorem *If the functional equations of zeta functions (2) of (G, ρ, V) hold, then the functional equations of zeta functions (2) of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ hold and vice versa.*

Proof. Put $\Omega(1) = \{\sigma \in \mathbb{C}^l \mid \xi(S(1), GL(n; k), \omega_{\sigma}) \neq 0\}$,

$\tilde{\Omega}(1) = \{\sigma \in \mathbb{C}^l \mid \xi(\tilde{S}(1), GL(m-n; k), \tilde{\omega}_{\sigma}) \neq 0\}$.

For a subset Ω' of \mathbb{C}^l , we denote $\mathbb{C}^l \setminus \Omega'$ by $C(\Omega')$.

Now from Proposition 1 of §2 and the fact that ω_{σ} is not trivial on $GL(n; k)$, we know that $C(\Omega(1))$ and $C(\tilde{\Omega}(1))$ are dense subsets of \mathbb{C}^l .

Now using the remarks above, we have proved the theorem. \square

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(注) 以上は 投稿予定の論文を、若干省略したものです。 応用等は省きましたが 例えば既約概均質ベクトル空間については、その少なくとも一つの k -form においては関数等式が成たつことがわかります。

また、以前 preprint を お渡ししたかたには、 (G_ρ, V) の関数等式から その castling transform について示すのでなく、 (G_ρ, V) において [S] の sufficient condition が成立するという仮定から その castling transform について 関数等式の成立を示していた事を 記しておきます (既約 p.v.については これで十分ですが)。